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# Decay Rates of the Derivatives of the Solutions of the Heat Equations and Related Topics

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## 1 Introduction

In this paper, we consider the initial-boundary value problem of the heat equation in the exterior domain of a ball,

$$(1.1) \quad \begin{cases} \frac{\partial}{\partial t} u = \Delta u - V(|x|)u & \text{in } \Omega_L \times (0, \infty), \\ \mu u + (1 - \mu) \frac{\partial}{\partial \nu} u = 0 & \text{on } \partial\Omega_L \times (0, \infty), \\ u(\cdot, 0) = \phi(\cdot) \in L^p(\Omega_L), \end{cases}$$

where  $0 \leq \mu \leq 1$ ,  $p \geq 1$ ,  $\Omega_L = \{x \in \mathbf{R}^N : |x| > L\}$ ,  $N \geq 2$ ,  $L > 0$ , and  $\nu$  is the outer unit normal vector to  $\partial\Omega_L$ . Throughout this paper, we assume that  $V = V(|x|)$  satisfies the following condition  $(V_\omega^\ell)$  for some  $\omega \geq 0$  and  $\ell \in \mathbf{N}$ :

$$(V_\omega^\ell) \quad \begin{cases} (0) & V = V(|x|) \in C^\ell(\mathbf{R}^N), \quad V \geq 0 \text{ in } \mathbf{R}^N, \\ (i) & \lim_{r \rightarrow \infty} r^2 V(r) = \omega, \\ (ii) & \int_L^\infty r \left| V(r) - \frac{\omega}{r^2} \right| dr < \infty, \\ (iii) & \sup_{r \geq L} \left| r^{2+j} \left( \frac{d^j}{dr^j} V \right) (r) \right| < \infty, \quad j = 1, \dots, \ell. \end{cases}$$

The purpose of this paper is to study the decay rates of the derivatives of the solution of (1.1) under the condition  $(V_\omega^\ell)$ , as  $t \rightarrow \infty$ .

Now, we introduce some notations. For any set  $A$  and  $B$ , let  $f = f(\lambda, \nu)$  and  $g = g(\lambda, \mu)$  be maps from  $A \times B$  to  $(0, \infty)$ . Then we say

$$f(\lambda, \mu) \preceq g(\lambda, \mu) \quad \text{for all } \lambda \in A$$

if, for any  $\mu \in B$ , there exists a positive constant  $C$  such that  $f(\lambda, \mu) \leq Cg(\lambda, \mu)$  for all  $\lambda \in A$ . Furthermore, we say

$$f(\lambda, \mu) \asymp g(\lambda, \mu) \quad \text{for all } \lambda \in A$$

if  $f(\lambda, \mu) \preceq g(\lambda, \mu)$  and  $g(\lambda, \mu) \preceq f(\lambda, \mu)$  for all  $\lambda \in A$ . We put

$$\mathbf{N}_0 = \mathbf{N} \cup \{0\}, \quad \mathbf{N}_0^N = \{(n_1, \dots, n_N) : n_i \in \mathbf{N}_0, i = 1, \dots, N\}.$$

Furthermore, for any  $j = (j_1, \dots, j_N) \in \mathbf{N}_0^N$ , we write  $|j| = \sum_{i=1}^N j_i$  and  $\nabla_x^j = \partial^{|j|} / \partial x_1^{j_1} \dots \partial x_N^{j_N}$ .

To state historical remarks, let  $\Omega$  be an unbounded domain in  $\mathbf{R}^N$ . Then, under the suitable assumptions on  $\Omega$  and  $V$ , for any  $j \in \mathbf{N}_0^N$ , the solution  $u$  of (1.1) in the domain  $\Omega$  satisfies

$$(1.2) \quad \|(\nabla_x^j u)(\cdot, t)\|_{L^\infty(\Omega)} \preceq t^{-\frac{N}{2p}} \|\phi\|_{L^p(\Omega)}$$

for all sufficiently large  $t$ . (See Theorem 10.1 of Chapters 3 and 4 in [6].) On the other hand, for the case when  $\Omega = \mathbf{R}^N$  (or  $\Omega = \mathbf{R}_+^N$ ) and  $V \equiv 0$ , the explicit representation of the fundamental solution of the heat equation implies that, for any  $j \in \mathbf{N}_0^N$ ,

$$(1.3) \quad \|(\nabla_x^j u)(\cdot, t)\|_{L^\infty(\mathbf{R}^N)} \preceq t^{-\frac{N}{2p} - \frac{|j|}{2}} \|\phi\|_{L^p(\mathbf{R}^N)}$$

for all  $t > 0$ . Furthermore, for the case when  $\Omega$  is a convex domain in  $\mathbf{R}^N$  and  $V \equiv 0$ , Li and Yau [7] studied the behavior of the nonnegative solution of (1.1) with  $\mu = 0$ , and obtained the inequality

$$(1.4) \quad \frac{|\nabla_x u|^2}{u^2} - \frac{\partial_t u}{u} \preceq \frac{1}{t}, \quad (x, t) \in \Omega \times (0, \infty).$$

Then, by the standard arguments in the parabolic equations, we see that, for any  $j \in \mathbf{N}_0^N$  with  $|j| \leq 1$ , the inequality (1.3) holds for all  $t > 0$ .

On the other hand, Grigor'yan and Saloff-Coste [2] studied the asymptotic behavior of the Green function  $G_\mu^V = G_\mu^V(x, y, t)$  of (1.1) for the case

when  $\Omega$  is the exterior domain of a compact set,  $\mu = 1$ , and  $V \equiv 0$ . They proved that, for any fixed  $x, y \in \Omega$ ,

$$G_1^V(x, y, t) \asymp t^{-\frac{N}{2}}$$

for all sufficiently large  $t$  if  $N \geq 3$ . This together with the mean value theorem, the Dirichlet boundary condition, and (1.2) implies that

$$\|(\nabla_x G_1^V)(\cdot, \cdot, t)\|_{L^\infty(\Omega \times \Omega)} \asymp t^{-\frac{N}{2}}$$

for all sufficiently large  $t$ . So we see that the solution of (1.1) with  $\mu = 1$  does not necessarily satisfy the inequality (1.3) even for the case  $|j| = 1$ . The first author of this paper studied the asymptotic behavior of the solution of the heat equation under the Neumann boundary condition in the exterior domain of a ball in [3]. His results imply that, for the case  $\mu = 0$  and  $V \equiv 0$  on  $\Omega_L$ , the inequality (1.3) does not necessarily hold for the case  $|j| = 2$ . Recently, Shibata and Shimizu [8] studied the decay properties of the Stokes semigroup in the exterior domain of a compact set, under the Neumann boundary condition. Their results are applicable to the heat equation, and we see that the inequality (1.3) holds for the case when  $N \geq 3$ ,  $\Omega$  is the exterior domain of a compact set,  $V \equiv 0$  on  $\Omega$ , and  $\mu = 0$ . Our motivation is how the decay rate is affected in the presence of  $V$  under various boundary conditions.

Let  $u_\mu^V = u_\mu^V(x, t : \phi)$  be a solution of the initial-boundary value problem (1.1) in the exterior domain  $\Omega_L$ . For any  $p \geq 1$  and  $t > 0$ , put

$$\|\nabla_x^j G_\mu^V(t)\|_{p \rightarrow \infty} = \sup \{ \|(\nabla_x^j u_\mu^V)(\cdot, t : \phi)\|_{L^\infty(\Omega_L)} : \|\phi\|_{L^p(\Omega_L)} = 1 \},$$

where  $j \in \mathbf{N}_0^N$ .

Let  $\Delta_{\mathbf{S}^{N-1}}$  be the Laplace-Beltrami operator on  $\mathbf{S}^{N-1}$  and  $\{\omega_k\}_{k=0}^\infty$  the eigenvalues of

$$(1.5) \quad -\Delta_{\mathbf{S}^{N-1}} Q = \omega_k Q \quad \text{on} \quad \mathbf{S}^{N-1}, \quad Q \in L^2(\mathbf{S}^{N-1}),$$

that is,

$$(1.6) \quad \omega_k = k(N + k - 2), \quad k \in \mathbf{N}_0.$$

Furthermore, let  $\{Q_{k,i}\}_{i=1}^{l_k}$  and  $l_k$  be the orthonormal system and the dimension of the eigenspace corresponding to  $\omega_k$ , respectively. Let  $U_{\mu,L}^V(r)$  be a solution of the initial value problem for the ordinary differential equation,

$$(O_V) \quad \begin{cases} \partial_r^2 U + \frac{N-1}{r} \partial_r U - V(r)U = 0 & \text{in } (L, \infty), \\ (\partial_r U)(L) = \mu, \quad U(L) = 1 - \mu, \end{cases}$$

where  $0 \leq \mu \leq 1$ . Put

$$(1.7) \quad g(t : \omega) = (1 + t)^{-\frac{\alpha(\omega)}{2}}.$$

Here  $\alpha = \alpha(\omega)$  is a nonnegative root of the equation  $\alpha(\alpha + N - 2) = \omega$ , that is,

$$(1.8) \quad \alpha(\omega) = \frac{-(N - 2) + \sqrt{(N - 2)^2 + 4\omega}}{2}.$$

Then, under the condition  $(V_\omega^1)$ , we see that

$$g(t : \omega) \asymp [U_{\mu, L}^V(t^{1/2})]^{-1}$$

for all sufficiently large  $t$  (see Proposition 2.1).

Now, we give the main results of this paper for the case  $N \geq 3$ .

**THEOREM 1.1** *Let  $N \geq 3$  and consider the initial-boundary value problem (1.1) under the condition  $(V_\omega^\ell)$  with  $\omega \geq 0$  and  $\ell \in \mathbf{N}$ . Let  $p \geq 1$ . Assume either*

$$(1.9) \quad \mu \neq \frac{2n'}{2n' + L} \quad \text{or} \quad V(r) \not\equiv \frac{\omega_{2n'}}{r^2} \quad \text{on} \quad [L, \infty)$$

for any  $n' \in \mathbf{N}_0$  with  $2n' \leq \ell + 1$ . Then, for any  $j \in \mathbf{N}_0^N$  with  $|j| \leq \ell + 1$ ,

$$(1.10) \quad \|\nabla_x^j G_\mu^V(t)\|_{p \rightarrow \infty} \leq t^{-\frac{N}{2p} - \frac{|j|}{2}} \quad \text{if} \quad |j| \leq \alpha(\omega),$$

$$(1.11) \quad \|\nabla_x^j G_\mu^V(t)\|_{p \rightarrow \infty} \asymp t^{-\frac{N}{2p} - \frac{\alpha(\omega)}{2}} \quad \text{if} \quad |j| > \alpha(\omega)$$

for all sufficiently large  $t$ .

If, for some  $n' \in \mathbf{N}_0$ , the equalities hold in (1.9), we have another decay property.

**THEOREM 1.2** *Let  $N \geq 3$  and consider the initial-boundary value problem (1.1). Assume that there exists a natural number  $n'$  such that*

$$(1.12) \quad n = 2n', \quad V(r) \equiv \frac{\omega_n}{r^2} \quad \text{on} \quad [L, \infty), \quad \mu = \frac{n}{n + L}.$$

Let  $p \geq 1$ . Then, for any  $j \in \mathbf{N}_0^N$ ,

$$(1.13) \quad \|\nabla_x^j G_\mu^V(t)\|_{p \rightarrow \infty} \leq t^{-\frac{N}{2p} - \frac{|j|}{2}} \quad \text{if} \quad |j| \leq n,$$

$$(1.14) \quad \|\nabla_x^j G_\mu^V(t)\|_{p \rightarrow \infty} \asymp t^{-\frac{N}{2p} - \frac{\alpha(\omega_n + \omega_1)}{2}} \quad \text{if} \quad |j| > n$$

for all sufficiently large  $t$ .

Here we remark that, under the condition (1.12),  $V$  satisfies the condition  $(V_\omega^\ell)$  for all  $\ell \in \mathbf{N}$  and  $\alpha(\omega) = \alpha(\omega_n) = n$ . Furthermore, as a corollary of Theorems 1.1 and 1.2, we have

**COROLLARY 1.1** *Let  $N \geq 3$  and  $u_\mu^V = u_\mu^V(x, t : \phi)$  be a solution of the initial-boundary value problem (1.1) with  $\phi \in L^p(\Omega_L)$ , under the condition  $(V_\omega^\ell)$  with  $\omega \geq 0$  and  $\ell \in \mathbf{N}$ . Let  $p \geq 1$  and  $j \in \mathbf{N}_0^N$  with  $|j| \leq \ell + 1$ . Then there exist positive constants  $C$  and  $T$  such that*

$$\|(\nabla_x^j u_\mu^V)(\cdot, t : \phi)\|_{L^\infty(\Omega_L)} \leq Ct^{-\frac{N}{2p} - \frac{|j|}{2}} \|\phi\|_{L^p(\Omega_L)}$$

for all  $t \geq T$  and all  $\phi \in L^p(\Omega_L)$  if and only if, either  $\omega \geq \omega_{|j|}$  or

$$|j| = 1, \quad V(r) \equiv 0 \quad \text{on} \quad [L, \infty), \quad \mu = 0.$$

According to Corollary 1.1, we may say that results in [7] and [8] are exceptional cases.

For the decay rates of the derivatives of the solution for case  $N = 2$ , similar results and peculiar results are both obtained although we will not give any proofs to the results for  $N = 2$ .

We first consider the cases either

$$(1.15) \quad N = 2 \quad \text{and} \quad \omega > 0$$

or

$$(1.16) \quad N = 2, \quad \mu = 0, \quad \text{and} \quad V \equiv 0 \quad \text{on} \quad [L, \infty).$$

**THEOREM 1.3** *Assume either (1.15) or (1.16). Then Theorems 1.1 and 1.2 hold true.*

Next, we consider the cases either

$$(1.17) \quad (N, \omega) = (2, 0) \quad \text{and} \quad \mu > 0$$

or

$$(1.18) \quad (N, \omega, \mu) = (2, 0, 0) \quad \text{and} \quad V \not\equiv 0 \quad \text{on} \quad [L, \infty).$$

Then we see that

$$U_{\mu, L}^0(r) = 1 - \mu + \mu \log \left( \frac{r}{L} \right).$$

**THEOREM 1.4** *Let  $N = 2$  and consider the initial-boundary value problem (1.1) under the condition  $(\tilde{V}_\omega^\ell)$  with  $\omega = 0$  and  $\ell \in \mathbf{N}$ . Let  $p \geq 1$ , and  $R > L$ . Assume either (1.17) or (1.18). Then, for any  $j \in \mathbf{N}_0^N$  with  $|j| \leq \ell + 1$ ,*

$$\begin{aligned} \|\nabla_x^j G_\mu^V(t)\|_{p \rightarrow \infty} &\asymp \|\nabla_x^j G_\mu^V(t)\|_{R:p \rightarrow \infty} \asymp t^{-\frac{1}{p}}(\log t)^{-1}, \\ \|r^{-|j|} \nabla_\theta^j G_\mu^V(t)\|_{p \rightarrow \infty} &\leq t^{-\frac{1}{p} - \frac{|j|}{2}} \end{aligned}$$

for all sufficiently large  $t$ .

In Section 2, we give fundamental lemmas and propositions without proofs. For their proofs, readers consult Sections 2 and 3 of [5]. Section 3 is devoted to the large time behavior of a radial solution to (1.1) with a radial initial value and its derivatives. Upper estimates for proofs of Theorems 1.1 and 1.2 are given in Section 4 and their proofs are provided in Section 5. As concluding remarks, some related topics are stated in Section 6.

## 2 Preliminaries

In this section, we give preliminary lemmas, whose proofs can be seen in Section 2 of [5], in order to study the decay rates of the derivatives of the solution (1.1) for the case  $N \geq 3$ .

For any  $\mu \in [0, 1]$ ,  $R \geq L$ , and  $\omega \geq 0$ , let  $U_{\mu,R}^\omega$  be the solution of

$$(O_\omega) \quad \begin{cases} \partial_r^2 U + \frac{N-1}{r} \partial_r U - \frac{\omega}{r^2} U = 0 & \text{in } (R, \infty), \\ (\partial_r U)(R) = \mu, \quad U(R) = 1 - \mu. \end{cases}$$

Put

$$(2.1) \quad U_+^\omega(r) = \left(\frac{r}{L}\right)^{\alpha(\omega)}, \quad U_-^\omega(r) = \left(\frac{r}{L}\right)^{-\beta(\omega)},$$

where  $\beta(\omega) = N - 2 + \alpha(\omega)$ . Then the functions  $U_+^\omega(r)$  and  $U_-^\omega(r)$  are solutions of the ordinary differential equation

$$(2.2) \quad \partial_r^2 U + \frac{N-1}{r} \partial_r U - \frac{\omega}{r^2} U = 0 \quad \text{in } (0, \infty),$$

and  $U_+^\omega(r) \not\equiv U_-^\omega(r)$  on  $(0, \infty)$ . So, by the uniqueness of the solution of  $(O_\omega)$ , there exist constants  $c_1$  and  $c_2$  such that

$$U_{\mu,R}^\omega(r) = c_1 U_+^\omega(r) + c_2 U_-^\omega(r), \quad r \geq R.$$

Therefore, by  $U_{\mu,R}^\omega(R) = 1 - \mu$  and  $\partial_r U_{\mu,R}^\omega(R) = \mu$ , we obtain

$$(2.3) \quad U_{\mu,R}^\omega(r) = \frac{\alpha - \mu\alpha - R\mu}{\alpha + \beta} \left(\frac{r}{R}\right)^{-\beta} + \frac{R\mu - \beta\mu + \beta}{\alpha + \beta} \left(\frac{r}{R}\right)^\alpha$$

where  $\alpha = \alpha(\omega)$  and  $\beta = \beta(\omega)$ . In what follows, we put

$$U_{\mu,R}^{\omega,k}(r) = U_{\mu,R}^{\omega+\omega_k}(r), \quad U_+^{\omega,k}(r) = U_+^{\omega+\omega_k}(r), \quad U_-^{\omega,k}(r) = U_-^{\omega+\omega_k}(r),$$

for simplicity. Then we have the following lemma on  $U_{\mu,R}^\omega$ .

**LEMMA 2.1** *Let  $L \leq R < S$  and  $a, b \geq 0$ . Assume  $N \geq 3$ . Then*

$$(2.4) \quad U_{\mu,R}^{a,k}(r) \asymp U_{\mu,R}^{b,k}(r)$$

for all  $r \in [R, S]$ ,  $\mu \in [0, 1]$ , and  $k \in \mathbf{N}_0$ ,

$$(2.5) \quad U_{\mu,R}^{a,k}(r) \asymp \left[ \frac{\mu}{k+1} + 1 - \mu \right] \left(\frac{r}{R}\right)^{\alpha(a+\omega_k)}$$

for all  $r \geq S$ ,  $\mu \in [0, 1]$ , and  $k \in \mathbf{N}_0$ , and

$$(2.6) \quad U_{0,R}^{a,k}(r) \asymp U_+^{a,k}(r)$$

for all  $r \geq R$  and  $k \in \mathbf{N}_0$ . Furthermore

$$(2.7) \quad 0 \leq \frac{d}{dr} U_{\mu,R}^{a,k}(r) \leq \frac{\mu + (k+1)(1-\mu)}{R} \left(\frac{r}{R}\right)^{\alpha(a+\omega_k)-1},$$

$$(2.8) \quad 0 < U_{\mu,R}^{a,k}(r) \leq \left[ \frac{\mu}{k+1} + 1 - \mu \right] \left(\frac{r}{R}\right)^{\alpha(a+\omega_k)},$$

for all  $r > R$ ,  $0 \leq \mu \leq 1$ , and  $k \in \mathbf{N}_0$ .

Next we recall the following two lemmas on the decay rate of the solutions of the initial-boundary value problem (1.1) under the condition  $(V_\omega^\ell)$ .

**LEMMA 2.2** *Let  $u_\mu^V$  be a solution of (1.1) under the condition  $(V_\omega^1)$  with  $\omega \geq 0$ . Let  $1 \leq p \leq q \leq \infty$  and  $i = 1, 2, \dots$ . Then there exists a positive constant  $C$ , independent of  $V$ , such that*

$$(2.9) \quad \|u_\mu^V(\cdot, t)\|_{L^q(\Omega_L)} \leq Ct^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})} \|\phi\|_{L^p(\Omega_L)}$$

for all  $t > 0$ .



**LEMMA 2.3** *Let  $u_\mu^V$  be a solution of (1.1) under the condition  $(V_\omega^\ell)$  with  $\omega \geq 0$  and  $\ell \geq 1$ . Then, for any  $\epsilon \in (0, 1)$  and  $p \geq 1$ , there exists a positive constant  $C$  such that*

$$(2.10) \quad |(\partial_t^i \nabla_x^j u_\mu^V)(x, t)| \leq C t^{-\frac{N}{2p} - \frac{|j|}{2} - i} \|\phi\|_{L^p(\Omega_L)},$$

for all  $(x, t) \in \Omega_L \times (0, \infty)$  with  $|x| \geq \epsilon t^{1/2} > L + 2$  and all  $i \in \mathbf{N}_0$  and  $j \in \mathbf{N}_0^N$  with  $2i + |j| \leq \ell + 1$ .

Next, we study the behavior of the solution  $U_{\mu,L}^V(r)$  of  $(O_V)$  under the assumption  $(V_\omega^\ell)$ . Put

$$V_k(r) = V(r) + \frac{\omega_k}{r^2}, \quad k \in \mathbf{N}_0.$$

In what follows, for  $k \in \mathbf{N}_0$  and  $\lambda \in \mathbf{R}$ , we put

$$\alpha_k = \alpha(\omega + \omega_k), \quad \beta_k = N - 2 + \alpha_k, \quad h_\lambda(r) = V(r) - \frac{\lambda}{r^2}$$

for simplicity. We first prove the following lemma.

**LEMMA 2.4** *Let  $R \geq L$ ,  $a \geq 0$ , and  $k \in \mathbf{N}_0$ . For any  $g \in C([R, \infty))$ , put*

$$H_R^{a,k}[g](r) = U_-^{a,k}(r) \int_R^r s^{1-N} [U_-^{a,k}(s)]^{-2} \left( \int_R^s \tau^{N-1} U_-^{a,k}(\tau) g(\tau) d\tau \right) ds.$$

Then

(i)  $H_R^{a,k}[g](r)$  is a solution of the ordinary differential equation

$$U'' + \frac{N-1}{r} U' - \frac{a + \omega_k}{r^2} U = g \quad \text{in } (R, \infty),$$

with  $U(R) = U'(R) = 0$ . In particular,

$$U_{\mu,R}^{V_k}(r) = U_{\mu,R}^{a,l}(r) + H_R^{a,l}[h_{\omega_l+a-\omega_k} U_{\mu,R}^{V_k}](r)$$

for all  $r \geq R$ ,  $k \in \mathbf{N}_0$ , and  $l = 0, \dots, k$ .

(ii) If  $g(r) \geq 0$  on  $[R, R_1]$  with  $R_1 > R$ , then

$$(2.11) \quad H_R^{a,k}[g](r) \geq 0, \quad H_R^{a,k}[g]'(r) \geq 0, \quad R \leq r \leq R_1.$$

(iii) Assume that there exists a positive constant  $A$  such that

$$(2.12) \quad |g(r)| \leq A |h_a(r)| U_{\mu,R}^{a,k}(r), \quad r \geq R.$$

Then there exist positive constants  $C_1$  and  $C_2$ , independent of  $R$  and  $k$ , such that

$$(2.13) \quad |H_R^{a,k}[g]'(r)| \leq C_1 A r^{-1} U_{\mu,R}^{a,k}(r) \int_R^r \tau |h_a(\tau)| d\tau,$$

$$(2.14) \quad |H_R^{a,k}[g](r)| \leq C_2 A U_{\mu,R}^{a,k}(r) \int_R^r \tau |h_a(\tau)| d\tau$$

for all  $r \geq R$ .

In view of Lemma 2.4, we have the following proposition on the behavior of  $U_{\mu,L}^{V_k}(r)$  as  $r \rightarrow \infty$ , by using the function  $U_{\mu,L}^{\omega,k}(r) = U_{\mu,L}^{\alpha(\omega+\omega_k)}(r)$ .

PROPOSITION 2.1 Assume  $(V_\omega^1)$  with  $\omega \geq 0$  and  $N \geq 3$ . Then

$$(2.15) \quad 0 \leq (\partial_r U_{\mu,L}^{V_k})(r) \leq (k+1) \left(\frac{r}{L}\right)^{\alpha_k-1}$$

for all  $r > L$ ,  $0 \leq \mu \leq 1$ , and  $k \in \mathbf{N}_0$ . Furthermore

$$(2.16) \quad U_{\mu,L}^{V_k}(r) \asymp U_{\mu,L}^{\omega,k}(r), \quad 0 \leq \mu \leq 1,$$

$$(2.17) \quad U_{0,L}^{V_k}(r) \asymp U_+^{\omega,k}(r)$$

for all  $r \geq L$  and  $k \in \mathbf{N}_0$ . In particular,

$$(2.18) \quad U_{\mu,L}^{V_k}(r) \asymp \left[ \frac{\mu}{k+1} + 1 - \mu \right] U_+^{\omega,k}$$

for all sufficiently large  $r$ ,  $0 \leq \mu \leq 1$ , and  $k \in \mathbf{N}_0$ .

Furthermore, by Proposition 2.1, we have the following proposition.

PROPOSITION 2.2 Assume  $(V_\omega^1)$  with  $\omega \geq 0$  and  $N \geq 3$ . For any  $g \in C([L, \infty))$ , put

$$F_L^V[g](r) = U_{0,L}^V(r) \int_L^r s^{1-N} [U_{0,L}^V(s)]^{-2} \left( \int_L^s \tau^{N-1} U_{0,L}^V(\tau) g(\tau) d\tau \right) ds.$$

Then, for any  $k \in \mathbf{N}_0$ ,  $F_L^{V_k}[g](r)$  is a solution of

$$(2.19) \quad \begin{cases} U'' + \frac{N-1}{r} U' - V_k(r) U = g & \text{in } (L, \infty), \\ U(L) = U'(L) = 0. \end{cases}$$

If there exist constants  $A > 0$  such that

$$|g(r)| \leq AU_{0,L}^{V_k}(r), \quad r \geq L,$$

then there exists a positive constant  $C$ , independent of  $k$ , such that

$$(2.20) \quad |F_L^{V_k}[g](r)| \leq CA(k+1)^{-1}r^2U_{0,L}^{V_k}(r),$$

$$(2.21) \quad |F_L^{V_k}[g]'(r)| \leq CArU_{0,L}^{V_k}(r),$$

for all  $r \geq L$ .

Next, we consider the case (1.12).

**PROPOSITION 2.3** Assume  $(V_\omega^\ell)$  with  $\omega \geq 0$  and  $\ell \in \mathbf{N}$ . Furthermore assume that there exists a multi-index  $J \in \mathbf{N}_0^N$  with  $|J| = n+1 \leq \ell+2$  such that

$$(2.22) \quad \begin{aligned} (\nabla_x^j U_{\mu,L}^V)(|x|) &\not\equiv 0 \text{ in } \Omega_L, \text{ for all } j \in \mathbf{N}_0^N \text{ with } |j| \leq n, \\ (\nabla_x^J U_{\mu,L}^V)(|x|) &\equiv 0 \text{ in } \Omega_L. \end{aligned}$$

Then there exists a nonnegative integer  $n'$  such that (1.12),

$$(2.23) \quad U_{\mu,L}^V(|x|) = \frac{1-\mu}{L^n}(x_1^2 + \cdots x_N^2)^{n'} = \frac{1-\mu}{L^n}|x|^n, \quad x \in \Omega_L,$$

and

$$(2.24) \quad (\nabla_x^j U_{\mu,L}^V)(|x|) \equiv 0 \text{ in } \Omega_L$$

hold for all  $j \in \mathbf{N}_0^N$  with  $|j| \geq n+1$ .

### 3 Derivatives of the solutions of $(P_\mu^k)$

In this section, we consider the radial solution  $v$  of the initial-boundary value problem

$$(P_\mu^k) \quad \begin{cases} \partial_t v = \Delta v - V_k(|x|)v & \text{in } \Omega_L \times (0, \infty), \\ \mu v - (1-\mu)\partial_r v = 0 & \text{on } \partial\Omega_L \times (0, \infty), \\ v(\cdot, 0) = \psi(\cdot) \in L^p(\Omega_L), \end{cases}$$

where  $0 \leq \mu \leq 1$ ,  $p \geq 1$ ,  $k \in \mathbf{N}_0$ , and  $\psi$  is a radial function in  $\Omega_L$ . For any positive  $\epsilon$  and  $T$ , put

$$\begin{aligned} D_\epsilon(T) &= \left\{ (x, t) \in \Omega_L \times (T, \infty) : |x| < \epsilon(1+t)^{1/2} \right\}, \\ \Gamma_\epsilon(T) &= \left\{ (x, t) \in \Omega_L \times (T, \infty) : |x| = \epsilon(1+t)^{1/2} \right\} \\ &\quad \cup \left\{ (x, T) : x \in \Omega_L, |x| \leq \epsilon(1+T)^{1/2} \right\}. \end{aligned}$$

We will construct a super-solution of  $(P_\mu^k)$  in  $D_\epsilon(T)$  for some positive constants  $\epsilon$  and  $T$ , and give some estimates on the derivatives of the solution  $v_\mu^k$  of  $(P_\mu^k)$  in  $D_\epsilon(T)$ . In what follows, under the assumption  $(V_\omega^\ell)$ , we put

$$U_k(r) = U_{0,L}^{V_k}(r), \quad g_k(t) = g(t : \omega + \omega_k)$$

for simplicity. We first construct a super-solution of  $(P_\mu^k)$ .

**LEMMA 3.1** *Assume  $N \geq 3$  and  $(V_\omega^\ell)$  with  $\omega \geq 0$  and  $k \in \mathbf{N}_0$ . Let  $\gamma > 0$ . Then there exist positive constants  $T$ ,  $\epsilon$ , and  $C$ , which are independent of  $k$ , and a function  $W = W(x, t)$  in  $\Omega_L \times (0, \infty)$  such that*

$$(3.1) \quad \partial_t W \geq \Delta W - V_k(|x|)W \quad \text{in } D_\epsilon(T),$$

$$(3.2) \quad \mu W(x, t) + (1 - \mu) \frac{\partial}{\partial \nu} W(x, t) \geq 0 \quad \text{on } \partial\Omega_L \times (T, \infty),$$

$$(3.3) \quad W(x, t) \geq C^{-\alpha_k} (1 + t)^{-\gamma} \quad \text{on } \Gamma_\epsilon(T),$$

and

$$(3.4) \quad 0 < W(x, t) \leq (1 + t)^{-\gamma} g_k(t) U_k(|x|) \quad \text{in } D_\epsilon(T).$$

**PROOF.** Let  $A$  and  $\epsilon$  be constants to be chosen later such that  $A > 0$  and  $0 < \epsilon < 1$ . Let  $T_\epsilon$  be a positive constant such that  $\epsilon(1 + T_\epsilon)^{1/2} = L + 1$ . Put

$$W(x, t) = (1 + t)^{-\gamma} g_k(t) \left[ U_k(|x|) - A(1 + k)(1 + t)^{-1} F_L^{V_k}[U_k](|x|) \right]$$

for all  $(x, t) \in \Omega_L \times (T_\epsilon, \infty)$ . Then, there exists a constant  $C_1 = C_1(\gamma)$  such that

$$(3.5) \quad \begin{aligned} \partial_t W &\geq [-\gamma(1 + t)^{-\gamma-1} g_k(t) + (1 + t)^{-\gamma} g'_k(t)] U_k(|x|) \\ &\geq -C_1(1 + k)(1 + t)^{-\gamma-1} g_k(t) U_k(|x|) \end{aligned}$$

and by (2.19), we have

$$(3.6) \quad \Delta W - V_k(|x|)W = -A(1 + k)(1 + t)^{-\gamma-1} g_k(t) U_k(|x|)$$

in  $\Omega_L \times (T_\epsilon, \infty)$ . Let  $A = C_1$ . Then, by (3.5) and (3.6), we have

$$(3.7) \quad \partial_t W \geq \Delta W - V_k(|x|)W \quad \text{in } \Omega_L \times (T_\epsilon, \infty).$$

On the other hand, by Proposition 2.2, there exists a positive constant  $C_2$ , independent of  $\epsilon$ , such that

$$(3.8) \quad 0 \leq A(1 + k)(1 + t)^{-1} F_L^{V_k}[U_k](|x|) \leq C_2 A \epsilon U_k(|x|)$$

for all  $(x, t) \in D_\epsilon(T_\epsilon)$ . Let  $0 < \epsilon \leq \min\{1, 1/2C_2A\}$ . Then we have

$$(3.9) \quad \frac{1}{2}g_k(t)U_k(|x|) \leq (1+t)^\gamma W(x, t) \leq g_k(t)U_k(|x|)$$

for all  $(x, t) \in D_\epsilon(T_\epsilon)$ . Then, by the definition of  $W$ , we have

$$(3.10) \quad \mu W + (1-\mu)\frac{\partial}{\partial \nu}W = \mu W \geq 0 \quad \text{on} \quad \partial\Omega_L \times (0, \infty).$$

By Proposition 2.1 and (1.7), we see that

$$(3.11) \quad U_k(\epsilon(1+t)^{1/2}) \asymp U_{\mu, L}^{\omega, k}(\epsilon(1+t)^{1/2}) \asymp (k+1)^{-1} \left(\frac{\epsilon}{L}\right)^{\alpha_k} [g_k(t)]^{-1}$$

for all  $t \geq T_\epsilon$  and  $k \in \mathbf{N}_0$ . By (3.9) and (3.11), there exists a positive constant  $C_3$  such that

$$(3.12) \quad \begin{aligned} (1+t)^\gamma W(x, t) &\geq \frac{1}{2}g_k(t)U_k(|x|) = \frac{1}{2}g_k(t)U_k(\epsilon(1+t)^{1/2}) \\ &\geq C_3^{-1}(k+1)^{-1} \left(\frac{\epsilon}{L}\right)^{\alpha_k} \end{aligned}$$

for all  $(x, t) \in \Gamma_\epsilon(T_\epsilon)$  with  $t > T_\epsilon$ . Furthermore, by (2.15), (3.9), and  $\epsilon(1+T_\epsilon)^{1/2} = L+1$ , there exists a positive constant  $C_4$  such that

$$(3.13) \quad \begin{aligned} W(x, T_\epsilon) &\geq \frac{1}{2}(1+T_\epsilon)^{-\gamma-\frac{\alpha_k}{2}} U_k(L) = \frac{1}{2}(1+T_\epsilon)^{-\gamma} \left(\frac{\epsilon}{L+1}\right)^{\alpha_k} \\ &\geq C_4^{-\alpha_k} (1+T_\epsilon)^{-\gamma} \end{aligned}$$

for all  $(x, T_\epsilon) \in \Gamma_\epsilon(T_\epsilon)$  and  $k \in \mathbf{N}_0$ . By (3.7), (3.10), (3.12), and (3.13), we have (3.1)–(3.4), and the proof of Lemma 3.1 is complete.  $\square$

Next we give the following lemmas on the estimates of derivatives of  $v_\mu^k$ . First, we estimate  $v$  and its time derivatives.

**LEMMA 3.2** *Assume that  $\psi$  is a radial function in  $\Omega_L$  such that  $\|\psi\|_{L^p(\Omega_L)} = 1$  with  $p \geq 1$ . Let  $N \geq 3$  and  $v$  be a solution of  $(P_\mu^k)$  with  $v(\cdot, 0) = \psi(\cdot)$  under the condition  $(V_\omega^\ell)$  with  $\omega \geq 0$ . Put*

$$w(x, t) = F_L^{V_k}[(\partial_t v)(\cdot, t)](|x|).$$

*Then there exist positive constants  $T$ ,  $\epsilon$ , and  $\eta$ , independent of  $k$ , such that*

$$(3.14) \quad |\partial_t^i v(x, t)| \leq \eta^{\alpha_k} t^{-\frac{N}{2p}-i} g_k(t) U_+^{\omega, k}(|x|),$$

$$(3.15) \quad |\partial_t^i w(x, t)| \leq \eta^{\alpha_k} t^{-\frac{N}{2p}-1-i} g_k(t) |x|^2 U_+^{\omega, k}(|x|)$$

*for all  $(x, t) \in D_\epsilon(T)$  and all  $i \in \mathbf{N}_0$  with  $2i \leq \ell + 1$ .*

PROOF. Let  $i \in \mathbf{N}_0$  and put  $v_i = \partial_t^i v$ . Let  $T$  and  $\epsilon$  be positive constants given in Lemma 3.1. Let  $W$  be the function constructed in Lemma 3.1 with  $\gamma = N/2p + i$ . For any  $\eta_1 > 0$ , we put

$$\bar{v}_i(x, t) = \eta_1^{\alpha_k} W(x, t)$$

for all  $(x, t) \in D_\epsilon(T)$ . Then, taking a sufficiently large  $T$  and  $\eta_1$  if necessary, by Lemma 2.3, we have

$$|v_i(x, t)| \leq \bar{v}_i(x, t) \quad \text{on } \Gamma_\epsilon(T).$$

So, by the comparison principle, we have

$$|v_i(x, t)| \leq \bar{v}_i(x, t) \quad \text{in } D_\epsilon(T).$$

This inequality together with (2.8), (2.16), and (3.4) implies

$$|v_i(x, t)| \leq \eta_1^{\alpha_k} t^{-\frac{N}{2p}-i} g_k(t) U_k(|x|) \leq \eta_1^{\alpha_k} t^{-\frac{N}{2p}-i} g_k(t) U_+^{\omega, k}(|x|)$$

for all  $(x, t) \in D_\epsilon(T)$ , and we obtain the inequality (3.14). On the other hand, since

$$(3.16) \quad (\partial_t^i w)(x, t) = F_L^{V_k}[(\partial_t^{i+1} v)(\cdot, t)](|x|)$$

for all  $(x, t) \in \Omega_L \times (0, \infty)$ , by (2.17), (2.20) and (3.14), we have (3.15), and the proof of Lemma 3.2 is complete.  $\square$

Furthermore we have the following lemma on the time derivatives of  $\partial_r v$  and  $\partial_r w$ .

**LEMMA 3.3** *Assume the same assumptions as in Lemma 3.2. Then there exist positive constants  $T$ ,  $\eta$ , and  $\epsilon$ , independent of  $k$ , such that*

$$(3.17) \quad |\partial_t^i \partial_r v(x, t)| \leq \eta^{\alpha_k} t^{-\frac{N}{2p}-i} g_k(t) |x|^{-1} U_+^{\omega, k}(|x|),$$

$$(3.18) \quad |\partial_t^i \partial_r w(x, t)| \leq \eta^{\alpha_k} t^{-\frac{N}{2p}-1-i} g_k(t) |x| U_+^{\omega, k}(|x|)$$

for all  $(x, t) \in D_\epsilon(T)$  and all  $i \in \mathbf{N}_0$  with  $2i \leq \ell + 1$ .

PROOF. By (2.17), (2.21), (3.14), and (3.16), we have (3.18). So we prove (3.17). Put  $v_i = \partial_t^i v$  and  $w_i = \partial_t^i w$ . Then  $v_i$  and  $w_i$  satisfy

$$\partial_t v_i = \Delta w_i - V_k(|x|) w_i$$

by the definition of  $F_L^{V_k}$ . By the uniqueness of the initial value problem for the ordinary differential equation, there exists a function  $\zeta(t)$  in  $(0, \infty)$  such that

$$(3.19) \quad v_i(x, t) = \zeta(t) U_{\mu, L}^{V_k}(|x|) + w_i(x, t)$$

for all  $(x, t) \in \Omega_L \times (0, \infty)$ . Furthermore, by (2.17), (2.20), (3.14), (3.15), and (3.19), there exist constants  $C_1, C_2, T, \eta_1$ , and  $\epsilon$  such that

$$\begin{aligned} |\zeta(t)|U_k(\epsilon(1+t)^{1/2}) &\leq |v_i(x, t)| \Big|_{|x|=\epsilon(1+t)^{1/2}} + |w_i(x, t)| \Big|_{|x|=\epsilon(1+t)^{1/2}} \\ &\leq C_1 t^{-\frac{N}{2p}-i} + C_2 \eta_1^{\alpha_k} t^{-\frac{N}{2p}-i} g_k(t) U_+^{\omega, k}(\epsilon(1+t)^{1/2}) \end{aligned}$$

for all  $t \geq T$ . This together with (3.11) implies that there exists a constant  $\eta_2$  such that

$$(3.20) \quad |\zeta(t)| \leq \eta_2^{\alpha_k} t^{-\frac{N}{2p}-i} g_k(t), \quad t \geq T, \quad k \in \mathbf{N}_0.$$

In addition, by (2.15), (3.18), and (3.19), there exists a constant  $\eta_3$  such that

$$\begin{aligned} |(\partial_r v_i)(x, t)| &\leq |\zeta(t)| (\partial_r U_{\mu, L}^{V_k})(|x|) + |\partial_r w_i(|x|, t)| \\ &\leq \eta_3^{\alpha_k} t^{-\frac{N}{2p}-i} g_k(t) U_+^{\omega, k}(|x|) |x|^{-1} \end{aligned}$$

for all  $(x, t) \in D_\epsilon(T)$  and  $k \in \mathbf{N}_0$ . So we obtain (3.17), and the proof of Lemma 3.3 is complete.  $\square$

We give upper estimates on the spatio-temporal derivatives of  $v$  and  $w$  and its proof is done in the similar way to the proofs of Lemmas 3.2 and 3.3.

**LEMMA 3.4** *Assume the same assumptions as in Lemma 3.2. Then there exist positive constants  $T, \eta$ , and  $\epsilon$ , independent of  $k$ , such that*

$$(3.21) \quad |\partial_t^i \partial_r^j v(x, t)| \leq \eta^{\alpha_k} t^{-\frac{N}{2p}-i} g_k(t) |x|^{-j} U_+^{\omega, k}(|x|),$$

$$(3.22) \quad |\partial_t^i \partial_r^j w(x, t)| \leq \eta^{\alpha_k} t^{-\frac{N}{2p}-1-i} g_k(t) |x|^{2-j} U_+^{\omega, k}(|x|)$$

for all  $(x, t) \in D_\epsilon(T)$ ,  $i \in \mathbf{N}_0$  with  $2(i+1) \leq \ell+1$ , and  $j = 2, \dots, \ell+2$ .

Finally, we give estimates on the derivatives of  $v$  for the case (1.12).

**LEMMA 3.5** *Assume that  $\psi$  is a radial function such that  $\|\psi\|_{L^p(\Omega_L)} = 1$  with  $p \geq 1$ . Let  $v$  be the solution of  $(P_\mu^k)$  with  $v(\cdot, 0) = \psi(\cdot)$  and  $k = 0$ , under the condition (1.12). Then, for any  $j \in \mathbf{N}_0^N$  with  $|j| \geq n+1$  and  $i \in \mathbf{N}_0$ , there exist positive constants  $C, T$ , and  $\epsilon$  such that*

$$(3.23) \quad |\partial_t^i \nabla_x^j v(x, t)| \leq C t^{-\frac{N}{2p}-\frac{1}{2}-i-\frac{n}{2}}$$

for all  $(x, t) \in D_\epsilon(T)$ .

PROOF. By (1.12), we have

$$U_{\mu,L}^V(x) = c \left( \sum_{i=1}^N x_i^2 \right)^{n'}, \quad U_+^\omega(r) = \left( \frac{r}{L} \right)^n, \quad g(t : \omega) = (1+t)^{-\frac{n}{2}},$$

where  $n = 2n'$  and  $c$  is a positive constant. (See also Proposition 2.3). Put  $v_i(x, t) = \partial_t^i v(x, t)$  and  $w_i(x, t) = F_L^V[v_{i+1}](|x|)$ . Let  $j \in \mathbf{N}_0^N$  with  $|j| \geq n+1$ . Then  $\nabla_x^j U_{\mu,L}^V(|x|) \equiv 0$  in  $\Omega_L$ , and by (3.19), we have  $\nabla_x^j v_i(x, t) = \nabla_x^j w_i(x, t)$  for all  $(x, t) \in \Omega_L \times (0, \infty)$ . Therefore, by the radial symmetry of  $w_i$  and the inequality (3.22) with  $k = 0$ , there exist positive constants  $T$  and  $\epsilon$  such that

$$\begin{aligned} |(\nabla_x^j v_i)(x, t)| &\leq \sum_{m=1}^{|j|} \frac{|(\partial_r^m w_i)(x, t)|}{|x|^{|j|-m}} \leq t^{-\frac{N}{2p}-1-i-\frac{n}{2}} |x|^{n+2-|j|} \\ &\leq t^{-\frac{N}{2p}-1-i-\frac{n}{2}} |x| \leq t^{-\frac{N}{2p}-\frac{1}{2}-i-\frac{n}{2}} \end{aligned}$$

for all  $(x, t) \in D_\epsilon(T)$ , and the proof of lemma 3.5 is complete.  $\square$

**REMARK 3.1** If the  $L^p$ -norm of the initial value is not 1, then all the right-hand terms in the estimates in Lemmas 3.2, 3.3 and 3.4 must be multiplied by  $\|\psi\|_{L^p(\Omega_L)}$ .

## 4 Upper bounds of derivatives of solutions

In this section, we prove the following two propositions, which are mentioned in Section 1 as upper estimates, by using lemmas given in the previous sections.

**PROPOSITION 4.1** *Assume the same assumptions as in Theorem 1.1. Then, for any  $p \geq 1$  and  $j \in \mathbf{N}_0^N$  with  $|j| \leq \ell + 1$ ,*

$$(4.1) \quad \|\nabla_x^j G_\mu^V(t)\|_{p \rightarrow \infty} \leq t^{-\frac{N}{2p} - \frac{\min\{\alpha(\omega), |j|\}}{2}}$$

for all sufficiently large  $t$ .

**PROPOSITION 4.2** *Assume the same assumptions as in Theorem 1.2. Then, for any  $p \geq 1$  and  $j \in \mathbf{N}_0^N$  with  $|j| \geq n + 1$ ,*

$$(4.2) \quad \|\nabla_x^j G_\mu^V(t)\|_{p \rightarrow \infty} \leq t^{-\frac{N}{2p} - \frac{\alpha(\omega_n + \omega_1)}{2}}$$

for all sufficiently large  $t$ .



PROOF OF PROPOSITION 4.1. Let  $u_\mu^V$  be the solution of (1.1) with  $\phi \in C_0(\Omega_L)$ . By the same arguments as in [3] and [4],  $\phi$  can be expanded in the Fourier series, that is, there exist radial functions  $\{\phi_{k,i}\} \subset L^2(\Omega_L)$  such that

$$(4.3) \quad \phi(x) = \sum_{k=0}^{\infty} \sum_{i=1}^{l_k} \phi_{k,i}(|x|) Q_{k,i} \left( \frac{x}{|x|} \right) \quad \text{in } L^2(\Omega_L).$$

Let  $u_\mu^{k,i}$  be a solution of (1.1) with the initial data  $\phi_{k,i}(|x|) Q_{k,i}(x/|x|)$  and  $v_\mu^{k,i}$  a radial solution of  $(P_\mu^k)$  with the initial data  $\phi_{k,i}$ . By the uniqueness of the solution of (1.1), we see that

$$(4.4) \quad u_\mu^{k,i}(x, t) = v_\mu^{k,i}(x, t) Q_{k,i} \left( \frac{x}{|x|} \right), \quad (x, t) \in \Omega_L \times (0, \infty),$$

where  $k \in \mathbf{N}_0$  and  $i = 1, \dots, l_k$ . On the other hand, by the standard elliptic regularity theorem and  $\|Q_{k,i}\|_{L^2(\mathbf{S}^{N-1})} = 1$ , for any  $n \in \mathbf{N}$ , we have

$$(4.5) \quad \|Q_{k,i}\|_{C^{2n}(\mathbf{S}^{N-1})} \preceq (1 + \omega_k)^{n+1} \asymp (k+1)^{2n+2}$$

for all  $k \in \mathbf{N}_0$  and  $i = 1, \dots, l_k$ . Furthermore the eigenspace of  $\Delta_{\mathbf{S}^{N-1}}$  corresponding to  $\omega_\ell$  is spanned by the functions  $\nabla_x^j |x|$  for  $j \in \mathbf{N}_0^N$  with  $|j| = \ell$ , and we have

$$(4.6) \quad l_k \leq N^k.$$

By the orthogonality of  $\{Q_{k,i}\}_{k,i}$ , we have

$$(4.7) \quad \int_{\Omega_L} u_\mu^{k_1, i_1}(x, t) u_\mu^{k_2, i_2}(x, t) dx = 0$$

for all  $t \geq 0$  if  $(k_1, i_1) \neq (k_2, i_2)$ . On the other hand, for any  $t > 0$ ,

$$(4.8) \quad u_\mu^V(x, t) = \lim_{m \rightarrow \infty} \sum_{k=0}^m \sum_{i=1}^{l_k} v_\mu^{k,i}(x, t) Q_{k,i} \left( \frac{x}{|x|} \right)$$

holds uniformly for all  $x \in \Omega_L$ . Hence we have

$$\begin{aligned} \int_{\partial B(0, |x|)} u_\mu^V(x, t) Q_{k,i} \left( \frac{x}{|x|} \right) d\sigma &= v_\mu^{k,i}(x, t) \int_{\partial B(0, |x|)} \left| Q_{k,i} \left( \frac{x}{|x|} \right) \right|^2 d\sigma \\ &= |x|^{N-1} v_\mu^{k,i}(x, t) \end{aligned}$$

for all  $(x, t) \in \Omega_L \times (0, \infty)$ . Then, by (4.5) and the Jensen inequality, we have

$$|x|^{N-1} |v_\mu^{k,i}(x, t)|^p \preceq (k+1)^{2p} \int_{\partial B(0, |x|)} |u_\mu^V(x, t)|^p d\sigma$$

for all  $(x, t) \in \Omega_L \times (0, \infty)$  and  $k \in \mathbb{N}_0$ . So, by (2.9), we have

$$(4.9) \quad \begin{aligned} \|v_\mu^{k,i}(\cdot, t)\|_{L^p(\Omega_L)} &\preceq \left( \int_L^\infty r^{N-1} |v_\mu^{k,i}(r, t)|^p dr \right)^{1/p} \\ &\preceq (k+1)^2 \|u_\mu^V(\cdot, t)\|_{L^p(\Omega_L)} \preceq (k+1)^2 \|\phi\|_{L^p(\Omega_L)} \end{aligned}$$

for all  $t > 0$  and  $k \in \mathbb{N}_0$ .

Let  $j \in \mathbb{N}_0^N$  with  $|j| \leq \ell + 1$ . Let  $k \in \mathbb{N}$  and  $i = 1, \dots, l_k$ . By (1.6), (4.4), and (4.5), we have

$$(4.10) \quad |\nabla_x^j u_\mu^{k,i}(x, t)| \preceq (k+1)^{\ell+3} \sum_{m=0}^{|j|} \frac{|\partial_r^m v_\mu^{k,i}(x, t)|}{|x|^{|j|-m}}, \quad (x, t) \in \Omega_L \times (0, \infty).$$

Since  $D_{\epsilon_1}(T) \subset D_{\epsilon_2}(T)$  if  $\epsilon_1 \leq \epsilon_2$ , by Lemmas 3.2, 3.3, 3.4, Remark 3.1 and (4.9), there exist positive constants  $\eta_1, \eta_2, \eta_3, T_*$ , and  $\epsilon_*$  such that

$$\begin{aligned} \frac{|\partial_r^m v_\mu^{k,i}(x, t+t_0)|}{|x|^{|j|-m}} &\preceq \eta_1^{\alpha_k} t^{-\frac{N}{2p}} g_k(t) U_+^{\omega, k}(|x|) |x|^{-|j|} \|v_\mu^{k,i}(\cdot, t_0)\|_{L^p(\Omega_L)} \\ &\preceq (k+1)^2 \epsilon^{[\alpha_k - |j|] + \eta_3^{\alpha_k} t^{-\frac{N}{2p} - \frac{\min\{\alpha_k, |j|\}}{2}}} \|\phi\|_{L^p(\Omega_L)} \end{aligned}$$

for all  $(x, t) \in D_\epsilon(T_*)$  with  $0 < \epsilon \leq \epsilon_*$ ,  $t_0 > 0$ , and  $m = 0, 1, \dots, |j|$ , where  $\alpha_k = \alpha(\omega + \omega_k)$ . Letting  $t_0 \rightarrow 0$ , we obtain

$$\frac{|\partial_r^m v_\mu^{k,i}(x, t)|}{|x|^{|j|-m}} \preceq (k+1)^2 \epsilon^{[\alpha_k - |j|] + \eta_3^{\alpha_k} t^{-\frac{N}{2p} - \frac{\min\{\alpha_k, |j|\}}{2}}} \|\phi\|_{L^p(\Omega_L)}$$

for all  $(x, t) \in D_\epsilon(T_*)$  with  $0 < \epsilon \leq \epsilon_*$  and  $m = 0, 1, \dots, |j|$ . This inequality together with (4.10) implies that

$$(4.11) \quad |\nabla_x^j u_\mu^{k,i}(x, t)| \preceq (k+1)^{\ell+5} \epsilon^{[\alpha_k - |j|] + \eta_3^{\alpha_k} t^{-\frac{N}{2p} - \frac{\min\{\alpha_k, |j|\}}{2}}} \|\phi\|_{L^p(\Omega_L)}$$

for all  $(x, t) \in D_\epsilon(T_*)$  with  $0 < \epsilon \leq \epsilon_*$ . Let  $0 < \epsilon \leq \epsilon_*$  and  $T_\epsilon$  be a positive constant such that  $T_\epsilon > T_*$  and  $\epsilon(1 + T_\epsilon)^{1/2} \geq L + 2$ . By (4.11), taking a sufficiently small  $\epsilon$  if necessary, we see

$$(4.12) \quad |\nabla_x^j u_\mu^{k,i}(x, t)| \preceq \frac{1}{2^k N^k} t^{-\frac{N}{2p} - \frac{\min\{\alpha_k, |j|\}}{2}} \|\phi\|_{L^p(\Omega_L)}$$

for all  $(x, t) \in D_\epsilon(T_\epsilon)$ ,  $k \in \mathbb{N}$ , and  $i = 1, \dots, l_k$ . Similarly, for the case  $k = 0$ , we have

$$(4.13) \quad \begin{aligned} |\nabla_x^j u_\mu^{0,1}(x, t)| &= |\nabla_x^j v_\mu^{0,1}(x, t)| \preceq \sum_{m=1}^{|j|} \frac{|(\partial_r^m v_\mu^{0,1})(x, t)|}{|x|^{|j|-m}} \\ &\preceq t^{-\frac{N}{2p} - \frac{\min\{\alpha_0, |j|\}}{2}} \|\phi\|_{L^p(\Omega_L)} \end{aligned}$$

for all  $(x, t) \in D_\epsilon(T_\epsilon)$ . By (4.6), (4.12), and (4.13), we obtain

$$(4.14) \quad |(\nabla_x^j u_\mu^V)(x, t)| \leq \limsup_{m \rightarrow \infty} \sum_{k=0}^m \sum_{i=1}^{l_k} |(\nabla_x^j u_\mu^{k,i})(x, t)| \\ \leq t^{-\frac{N}{2p} - \frac{\min\{\alpha_0, |j|\}}{2}} \|\phi\|_{L^p(\Omega_L)}$$

for all  $(x, t) \in D_\epsilon(T_\epsilon)$ . On the other hand, by Lemma 2.3, we have

$$(4.15) \quad |(\nabla_x^j u_\mu^V)(x, t)| \leq t^{-\frac{N}{2p} - \frac{|j|}{2}} \|\phi\|_{L^p(\Omega_L)}$$

for all  $(x, t) \notin D_\epsilon(T_\epsilon)$ . Therefore, by (4.14) and (4.15), we obtain

$$(4.16) \quad |(\nabla_x^j u_\mu^V)(x, t)| \leq t^{-\frac{N}{2p} - \frac{\min\{\alpha(\omega), |j|\}}{2}} \|\phi\|_{L^p(\Omega_L)}$$

for all  $(x, t) \in \Omega_L$  with  $t \geq T_\epsilon$ , where  $\phi \in C_0(\Omega_L)$ . Since  $C_0(\Omega_L)$  is a dense subset of  $L^p(\Omega_L)$ , the inequality (4.16) holds for all  $\phi \in L^p(\Omega_L)$ , and the proof of Proposition 4.1 is complete.  $\square$

PROOF OF PROPOSITION 4.2. By (1.12),  $V$  satisfies the condition  $(V_\omega^\ell)$  with  $\omega = \omega_n$  and  $\ell = 0, 1, 2, \dots$ . Let  $j \in \mathbf{N}_0^N$  with  $|j| \geq n+1 = 2n'+1$ . Let  $u_\mu^V$  be the solution of (1.1) with  $\phi \in C_0(\Omega_L)$  and  $u_\mu^{k,i}$  a function given in the proof of Proposition 4.1. By the same argument as in the proof of (4.13) and Lemma 3.5, for any sufficiently small  $\epsilon > 0$ , there exists a positive constant  $T_\epsilon$  such that

$$(4.17) \quad |(\nabla_x^j u_\mu^{0,1})(x, t)| \leq t^{-\frac{N}{2p} - \frac{n+1}{2}} \|\phi\|_{L^p(\Omega_L)}$$

for all  $(x, t) \in D_\epsilon(T_\epsilon)$ .

On the other hand, by the same argument as in the proof of (4.14), taking a sufficiently small  $\epsilon > 0$  if necessary, we have

$$(4.18) \quad \limsup_{m \rightarrow \infty} \sum_{k=1}^m \sum_{i=1}^{l_k} |(\nabla_x^j u_\mu^{k,i})(x, t)| \leq t^{-\frac{N}{2p} - \frac{\min\{\alpha(\omega_n + \omega_1), |j|\}}{2}} \|\phi\|_{L^p(\Omega_L)}$$

for all  $(x, t) \in D_\epsilon(T_\epsilon)$ . We note that  $\alpha(\omega_n + \omega_1) \leq \alpha(\omega_n) + 1 = n+1$ . Therefore, by (4.17), (4.18), and  $|j| \geq n+1$ , we have

$$(4.19) \quad |(\nabla_x^j u_\mu^V)(x, t)| \leq t^{-\frac{N}{2p} - \frac{\alpha(\omega_n + \omega_1)}{2}} \|\phi\|_{L^p(\Omega_L)}$$

for all  $(x, t) \in D_\epsilon(T_\epsilon)$ . Furthermore, by (4.15) and (4.19), taking a sufficiently small  $\epsilon$  if necessary, we have

$$(4.20) \quad |(\nabla_x^j u_\mu^V)(x, t)| \leq t^{-\frac{N}{2p} - \frac{\alpha(\omega_n + \omega_1)}{2}} \|\phi\|_{L^p(\Omega_L)}$$

for all  $(x, t) \in \Omega_L \times (T_\epsilon, \infty)$ , where  $\phi \in C_0(\Omega_L)$ . Furthermore, since  $C_0(\Omega_L)$  is a dense subset of  $L^p(\Omega_L)$ , we have the inequality (4.20) for all  $\phi \in L^p(\Omega_L)$ , and the proof of Proposition 4.2 is complete.  $\square$

## 5 Proofs of Theorems 1.1 and 1.2

In this section we consider the asymptotic behavior of the derivatives of the radial solution  $v$  of (1.1) for some initial data  $\psi \in C_0(\Omega_L)$  and complete proofs of Theorems 1.1 and 1.2.

**PROPOSITION 5.1** *Let  $R > 0$ ,  $\omega \geq 0$ , and  $\psi(\not\equiv 0)$  be a nonnegative, radial function belonging to  $C_0(\Omega_R)$ . Let  $v$  be a radial solution of*

$$(5.1) \quad \begin{cases} \partial_t v = \Delta v - \frac{\omega}{|x|^2} v & \text{in } \Omega_R \times (0, \infty), \\ v(x, t) = 0 & \text{on } \partial\Omega_R \times (0, \infty), \\ v(x, 0) = \psi(x) & \text{in } \Omega_R. \end{cases}$$

Then, for any  $p \in [1, \infty]$ ,

$$(5.2) \quad \|v(\cdot, t)\|_{L^p(\Omega_R)} \asymp t^{-\frac{N}{2}(1-\frac{1}{p})-\frac{\alpha(\omega)}{2}}$$

holds for all sufficiently large  $t$ . Furthermore there exists a positive constant  $\epsilon_*$  such that, for any  $0 < \epsilon \leq \epsilon_*$ ,

$$(5.3) \quad v(x, t) \Big|_{|x|=\epsilon(1+t)^{1/2}} \asymp \epsilon^{\alpha(\omega)} t^{-\frac{N+\alpha(\omega)}{2}}, \quad t > T$$

holds with suitably chosen  $T = T(\epsilon)$ .

**PROOF.** Put

$$(5.4) \quad z(y, s) = (1+t)^{\frac{N+\alpha}{2}} v(x, t), \quad y = (1+t)^{-\frac{1}{2}} x, \quad s = \log(1+t),$$

where  $\alpha = \alpha(\omega)$ . Then the function  $z$  satisfies

$$(5.5) \quad \begin{cases} \partial_s z = \frac{1}{\rho} \operatorname{div}(\rho \nabla_y z) + \frac{N+\alpha}{2} z - \frac{\omega}{|y|^2} z & \text{in } W, \\ z = 0 & \text{on } \partial W, \\ z(y, 0) = \psi(y) & \text{in } \Omega_R, \end{cases}$$

where  $\rho(y) = \exp(|y|^2/4)$  and

$$\Omega(s) = e^{-s/2} \Omega_R, \quad W = \bigcup_{0 < s < \infty} (\Omega(s) \times \{s\}), \quad \partial W = \bigcup_{0 < s < \infty} (\partial\Omega(s) \times \{s\}).$$

Put

$$\varphi(y) = c_0 |y|^{\alpha(\omega)} \exp(-|y|^2/4),$$

where  $c_0$  is a positive constant such that  $\|\varphi\|_{L^2(\mathbf{R}^N, \rho dy)} = 1$ . Then, since

$$\int_{\Omega_R} v(x, t) U_{1,R}^\omega(|x|) dx = \int_{\Omega_R} \phi(x) U_{1,R}^\omega(|x|) dx > 0, \quad t \geq 0,$$

by the same argument as in the proof of Lemma 6.1 in [4], we see that

$$(5.6) \quad a \equiv \int_{\Omega_R} \phi(x) U_{1,R}^\omega(|x|) dx = \lim_{s \rightarrow \infty} \int_{\Omega(s)} z(y, s) \varphi(y) \rho(y) dy > 0.$$

Furthermore, by the same argument as in the proof of Lemmas 3.3 and 3.4 in [4], for any  $r_1$  and  $r_2$  with  $0 < r_1 < r_2$ , we have

$$(5.7) \quad \sup_{s > 0} \|z(\cdot, s)\|_{L^2(\Omega(s), \rho dy)} < \infty,$$

$$(5.8) \quad \sup_{s > 0} \|z(\cdot, s)\|_{L^\infty(\{y : |y| \geq r_1\})} < \infty,$$

$$(5.9) \quad \lim_{s \rightarrow \infty} \|z(\cdot, s) - a\varphi\|_{C(\{y : r_1 \leq |y| \leq r_2\})} = 0.$$

By (5.6), (5.7) and (5.9), we have  $\|z(\cdot, s)\|_{L^1(\Omega(s))} \asymp 1$  for all sufficiently large  $s$ . So, by (5.8) and (5.9), for any  $p \in [1, \infty]$ , we have  $\|z(\cdot, s)\|_{L^p(\Omega(s))} \asymp 1$  for all sufficiently large  $s$ , and obtain (5.2).

On the other hand, by the same argument as in (3.19), there exists a function  $\zeta$  in  $(0, \infty)$  such that

$$(5.10) \quad v(x, t) = \zeta(t) U_{0,R}^V(|x|) + F_L^V[(\partial_t v)(\cdot, t)](|x|)$$

for all  $(x, t) \in \Omega_R \times (0, \infty)$  with  $V = \omega/r^2$ . By (5.2) with  $p = \infty$ , we may apply the same arguments as in the proof of Lemma 3.2 with  $\gamma = (N + \alpha(\omega))/2$  to  $v$ . Then we see that there exist positive constants  $\epsilon_*$  and  $T_*$  such that

$$(5.11) \quad |F_L^V[(\partial_t v)(\cdot, t)](|x|)| \preceq t^{-\frac{N}{2} - \alpha(\omega) - 1} |x|^{\alpha(\omega) + 2}$$

for all  $(x, t) \in D_{\epsilon_*}(T_*)$ . Therefore, by (2.18), (5.9), (5.10), (5.11), and the same arguments as in the deduction of (3.20), we may take a sufficiently small  $\tilde{\epsilon}$  so that

$$(5.12) \quad \begin{aligned} \zeta(t) &= [U_{0,R}^V(\tilde{\epsilon}(1+t)^{1/2})]^{-1} \left[ v(x, t) - F_L^V[(\partial_t v)(\cdot, t)](|x|) \right] \Big|_{|x|=\tilde{\epsilon}(1+t)^{1/2}} \\ &\asymp \tilde{\epsilon}^{-\alpha} t^{-\frac{\alpha}{2}} \left[ t^{-\frac{N+\alpha}{2}} + O(\tilde{\epsilon}^{\alpha+2}) t^{-\frac{N+\alpha}{2}} \right] \asymp t^{-\frac{N}{2} - \alpha} \end{aligned}$$

for all sufficiently large  $t$ . Then, by (5.10)–(5.12) and the similar argument as in (5.12), we have (5.3), and the proof of Proposition 5.1 is complete.  $\square$

PROOF OF THEOREM 1.1. Assume  $(V_\omega^\ell)$ . Let  $\tilde{\omega}$  be a constant such that  $\tilde{\omega} > \omega$  and

$$(5.13) \quad \alpha(\tilde{\omega}) < \alpha(\omega) + 1.$$

Then, by  $(V_\omega^\ell)$ -(i), we may take a sufficiently large  $R$  so that

$$V(r) \leq \frac{\tilde{\omega}}{r^2}, \quad r \geq R.$$

Let  $p \geq 1$  and  $\psi(\not\equiv 0)$  be a nonnegative, radial function belonging to  $C_0(\Omega_R)$ . Let  $v$  be a solution of (5.1) with  $\omega$  replaced by  $\tilde{\omega}$ . For any  $T > 0$ , let  $u_T^V$  be a solution of (1.1) with the initial data  $\phi(\cdot) = v(\cdot, T)/\|v(\cdot, T)\|_{L^p(\Omega_R)}$ . Here we remark that

$$(5.14) \quad \|u_T^V(\cdot, 0)\|_{L^p(\Omega_L)} = 1.$$

By the comparison principle, (5.2), and (5.3), for any sufficiently small  $\epsilon > 0$ , there exists a positive constant  $T_\epsilon$  such that

$$(5.15) \quad \begin{aligned} u_T^V(x, T) &\geq \frac{v(x, 2T)}{\|v(\cdot, T)\|_{L^p(\Omega_R)}} \asymp T^{\frac{N}{2}(1-\frac{1}{p})+\frac{\alpha(\tilde{\omega})}{2}} v(x, 2T) \\ &\succeq \epsilon^{\alpha(\tilde{\omega})} T^{-\frac{N}{2p}} \end{aligned}$$

for all  $(x, T) \in \Omega_L \times (T_\epsilon, \infty)$  with  $|x| = \epsilon(1 + 2T)^{1/2} > \max\{R, 2L + 2\}$ .

On the other hand, there exists a function  $\zeta_V(t)$  such that

$$(5.16) \quad u_T^V(x, t) = \zeta_V(t) U_{\mu, L}^V(|x|) + F_L^V[\partial_t u_T^V](|x|)$$

for all  $x \in \Omega_L$ . By Lemmas 3.2–3.4 and (5.14), taking a sufficiently small  $\epsilon$  and sufficiently large  $T_\epsilon$  if necessary, we have

$$(5.17) \quad |\partial_r^j F_L^V[\partial_t u_T^V](|x|)| \preceq t^{-\frac{N}{2p}-1-\frac{\alpha(\omega)}{2}} |x|^{2-|j|+\alpha(\omega)}$$

for all  $(x, t) \in D_\epsilon(T_\epsilon)$  and  $j \in \mathbb{N}_0$  with  $|j| \leq \ell + 2$ . Furthermore, by (5.13) and (5.15)–(5.17), there exist positive constants  $C_1$  and  $C_2$  such that

$$\begin{aligned} \zeta_V(T) U_{\mu, L}^V(|x|) &\geq u_T^V(x, T) - |F_L^V[\partial_t u_T^V](|x|)| \\ &\geq C_1 \epsilon^{\alpha(\tilde{\omega})} T^{-\frac{N}{2p}} - C_2 \epsilon^{\alpha(\omega)+2} T^{-\frac{N}{2p}} \succeq \epsilon^{\alpha(\omega)+1} T^{-\frac{N}{2p}} \end{aligned}$$

for all  $x \in \Omega_L$  with  $L + 1 < |x| = \epsilon(1 + 2T)^{1/2}/2 < \epsilon(1 + T)^{1/2}$  and  $T \geq T_\epsilon$ . Therefore, by (2.5) and (2.16), we have

$$(5.18) \quad \zeta_V(T) \succeq T^{-\frac{N}{2p}-\frac{\alpha(\omega)}{2}}$$

for all sufficiently large  $T$ . Therefore, by (5.16)–(5.18), there exist positive constants  $C_3$  and  $C_4$  such that

$$(5.19) \quad \begin{aligned} & |\nabla_x^j u_T^V(x, T)| \\ & \geq C_4 T^{-\frac{N}{2p} - \frac{\alpha(\omega)}{2}} |\nabla_x^j U_{\mu, L}^V(x)| - C_4 T^{-\frac{N}{2p} - 1 - \frac{\alpha(\omega)}{2}} |x|^{2+\alpha(\omega)-|j|} \end{aligned}$$

for all  $L < |x| \leq \epsilon(1+T)^{1/2}$ ,  $T \geq T_\epsilon$ , and  $j \in \mathbf{N}_0^N$  with  $|j| \leq \ell$ .

Let  $j \in \mathbf{N}_0^N$  with  $|j| \leq \ell$ . By the assumption of Theorem 1.1 and Proposition 2.3, there exists a point  $x_0 \in \Omega_L$  such that  $(\nabla_x^j U_{\mu, L}^V)(x_0) \neq 0$ . Then, by (5.19), there exist positive constants  $C_5$  and  $C_6$  such that

$$(5.20) \quad |(\nabla_x^j u_T^V)(x_0, T)| \geq C_5 T^{-\frac{N}{2p} - \frac{\alpha(\omega)}{2}} - C_6 T^{-\frac{N}{2p} - \frac{\alpha(\omega)}{2} - 1} \geq T^{-\frac{N}{2p} - \frac{\alpha(\omega)}{2}}$$

for all sufficiently large  $T$ . This inequality together with (5.14) implies

$$(5.21) \quad \|\nabla_x^j G_\mu^V(T)\|_{p \rightarrow \infty} \geq T^{-\frac{N}{2p} - \frac{\alpha(\omega)}{2}}$$

for all sufficiently large  $T$ . This together with Proposition 4.1 implies (1.10) and (1.11), and the proof of Theorem 1.1 is complete.  $\square$

PROOF OF THEOREM 1.2. Let  $u_T^{V_1}$  be a function given in the proof of Theorem 1.1 with  $V(r) = (\omega_n + \omega_1)/r^2$ . Put

$$\tilde{u}_T^V(x, t) = u_T^{V_1}(x, t) \frac{x_1}{|x|}.$$

Then  $\tilde{u}_T^V$  is a solution of (1.1) with  $V(r) = \omega_n/r^2$ .

Let  $j = (j_1, \dots, j_N) \in \mathbf{N}_0^N$  with  $|j| \geq n+1$ . Put  $j' = (j_1+1, j_2, \dots, j_N)$  and

$$\tilde{U}_{\mu, L}^{\omega_n + \omega_1}(r) = \int_L^r U_{\mu, L}^{\omega_n + \omega_1}(s) ds$$

Then, by (2.5), we see that  $\tilde{U}_{\mu, L}^{\omega_n + \omega_1}(r) \asymp r^{\alpha(\omega_n + \omega_1) + 1}$  for all sufficiently large  $r$ . If  $\nabla_x^{j'} \tilde{U}_{\mu}^{\omega_n + \omega_1}(|x|) \equiv 0$  in  $\Omega_L$ , then, we see that  $\tilde{U}_{\mu, L}^{\omega_n + \omega_1}(r)$  is a polynomial. This contradicts  $\alpha(\omega_n + \omega_1) \notin \mathbf{N}$  if  $n \geq 1$ . If  $n = 0$ , by (1.12),

$$U_{\mu, L}^{\omega_n + \omega_1}(r) = U_{0, L}^{\omega_1}(r) = \frac{1}{N} \left( \frac{r}{L} \right)^{-(N-1)} + \frac{N-1}{LN} r,$$

and  $\tilde{U}_{\mu}^{\omega_n + \omega_1}(r)$  is not a polynomial. So we have

$$\nabla_x^{j'} \tilde{U}_{\mu}^{\omega_n + \omega_1}(|x|) = \nabla_x^j \left[ U_{\mu}^{\omega_n + \omega_1}(|x|) \frac{x_1}{|x|} \right] \neq 0 \quad \text{in } \Omega_L.$$

By the similar arguments in (5.16)–(5.20) and  $\omega = \omega_n + \omega_1$ , there exist positive constants  $C_1$  and  $C_2$  such that

$$\begin{aligned} |(\nabla_x^j \tilde{u}_T^V)(x_0, T)| &\geq C_1 T^{-\frac{N}{2} - \frac{\alpha(\omega_n + \omega_1)}{2}} - C_2 T^{-\frac{N}{2} - \alpha(\omega_n + \omega_1) - 1} \\ &\succeq T^{-\frac{N}{2} - \frac{\alpha(\omega_n + \omega_1)}{2}} \end{aligned}$$

for all sufficiently large  $T$ . Furthermore, since  $\|\tilde{u}_T^V(\cdot, 0)\|_{L^p(\Omega_L)} \asymp 1$ , we obtain

$$\|\nabla_x^j G_\mu^V(T)\|_{p \rightarrow \infty} \succeq T^{-\frac{N}{2p} - \frac{\alpha(\omega_n + \omega_1)}{2}}$$

for all sufficiently large  $T$ . Therefore, this inequality together with Propositions 4.1 and 4.2 imply (1.13) and (1.14), and the proof of Theorem 1.2 is complete.  $\square$

## 6 Concluding remarks

As concluding remarks, we state some related topics. In the previous sections, we treat the exterior of a ball, however, we can treat the whole space and we can argue the movement of hot spots (the maximum points of a solution) with a potential  $V$ . According to the decay order of  $V$ , the behavior of hot spots varies. Such works are now in progress and we will discuss these topics later.

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